Statistical Inference for Diffusion Processes

Radu Herbei

The Ohio State University
The Mathematical Biosciences Institute

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Quick review

- $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathbb{W}_t, t \in [0, T]\}$ is a SBM, $\mathcal{F}_t$ - standard filtration
- If a stochastic process (h_t), $t \in [0, T]$ is carefully chosen, one can define the Itô integral

$$\int_0^T h_t \; d\mathbb{W}_t$$

- A homogeneous diffusion process (X_t) is characterized via

$$dX_t = S(X_t)dt + \sigma(X_t)d\mathbb{W}_t \quad X(0) = X_0, \; t \in [0, T]$$

- Weak solutions, existence, uniqueness, Girsanov formula.

What's next

- Numerical methods for SDEs
- Ergodicity, LLN, CLTs
- Statistical inference for SDEs (high frequency data)
- Statistical inference for SDEs (discrete data)

Numerical methods for SDEs

A “discretization” of a stochastic process $X^T = \{X_t, t \in [0, T]\}$ is a process X_δ^T, which “approximates” X^T in some way.

The discretization X_δ has *strong order of convergence* γ if for any $T > 0$,

$$E(|X_\delta(T) - X(T)|) \leq c\delta^\gamma \quad \text{for all} \quad \delta < \delta_0$$

The discretization X_δ^T has *weak order of convergence* β if for any function g which is $2(\beta + 1)$ continuously differentiable, it is true that

$$|E(g(X_\delta(T))) - E(g(X(T)))| \leq c\delta^\beta \quad \text{for all} \quad \delta < \delta_0$$

Euler scheme

$$dX_t = S(t, X_t)dt + \sigma(t, X_t)d\mathbb{W}_t$$

Given a collection of time points $0 = t_0 < t_1 \cdots < t_n = T$, the Euler discretization is the process $X^E(\cdot)$ such that

$$X^E(t_{i+1}) = X^E(t_i) + S(t_i, X^E(t_i))(t_{i+1} - t_i) + \sigma(t_i, X^E(t_i))(\mathbb{W}(t_{i+1)} - \mathbb{W}(t_i))$$

For $t \in [t_i, t_{i+1})$, the process $X^E(t)$ can be defined in any way – typically by linear interpolation.

The Euler scheme has strong order of convergence $\gamma = 1/2$ and weak order $\beta = 1$.

Euler scheme

- Most popular approach to approximate a diffusion process
- Vast majority of research using SDE models actually use the Euler approximation for the computing implementation
- Very easy to implement; very fast to simulate
  - Simulate $Z \sim \mathcal{N}(0,1)$
  - Given $X^E(t_i)$, set

$$X^E(t_{i+1}) = X^E(t_i) + S(t_i, X^E(t_i))(t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i}\,\sigma(t_i, X^E(t_i))Z$$

- With a little "trick", can achieve weak order of convergence $\beta = 1$.

Milstein scheme

$$
\begin{aligned}
&X^M(t_{i+1}) = \\
&= X^M(t_i) + S(t_i, X^M(t_i))(t_{i+1} - t_i) + \sigma(t_i, X^M(t_i))(\mathbb{W}(t_{i+1)} - \mathbb{W}(t_i)) \\
&+ \frac{1}{2}\sigma(t_i, X^M(t_i))\sigma_x(t_i, X^M(t_i))\Big((\mathbb{W}(t_{i+1}) - \mathbb{W}(t_i))^2 - (t_{i+1} - t_i)\Big)
\end{aligned}
$$

Milstein scheme has strong order of convergence $\gamma = 1$ and weak order of convergence $\beta = 1$.

Note that for the OU process, $\sigma_x \equiv 0$ thus, the Euler and Milstein schemes are identical.

Geometric BM

$$dX_t = \theta_1 X_t dt + \theta_2 X_t d\mathbb{W}_t$$

Euler scheme:

$$X_{i+1}^E = X_i^E(1 + \theta_1 \Delta) + \theta_2 X_i^E \sqrt{\Delta} Z = X_i^E(1 + \theta_1 \Delta + \theta_2 \sqrt{\Delta} Z)$$

Milstein scheme

$$\begin{aligned} X_{i+1}^M &= X_i^M + \theta_1 X_i^M \Delta + \theta_2 X_i^M \sqrt{\Delta} Z + \frac{1}{2}\theta_2^2 X_i^M (\Delta Z^2 - \Delta) \\ &= X_i^M \left(1 + \left(\theta_1 + \frac{1}{2}\theta_2^2 (Z^2 - 1)\right) \Delta + \theta_2 \sqrt{\Delta} Z \right) \end{aligned}$$

Exact solution

$$\begin{aligned} X_{t+\Delta} &= X_t \exp\left\{ \left(\theta_1 - \frac{\theta_2^2}{2}\right) \Delta + \theta_2 \sqrt{\Delta} Z \right\} \\ &= X_t \left(1 + \left(\theta_1 - \frac{\theta_2^2}{2}\right) \Delta + \theta_2 \sqrt{\Delta} Z + \frac{1}{2}\theta_2^2 \Delta Z^2 + \mathcal{O}(\Delta) \right) \end{aligned}$$

Connection between Euler and Milstein schemes

$$dX_t = S(t, X_t)dt + \sigma(t, X_t)d\mathbb{W}_t$$

Consider the Lamperti transformation $Y_t = F(X_t)$ where

$$F(x) = \int_z^x \frac{1}{\sigma(t, u)} du$$

Note that

$$F'(x) = \frac{1}{\sigma(t, x)} \qquad F''(x) = -\frac{\sigma_x(t, x)}{\sigma(t, x)^2}$$

Use Itô formula to derive that

$$dY_t = \left(\frac{S(t, X_t)}{\sigma(t, X_t)} - \frac{1}{2}\sigma_x(t, X_t) \right) dt + d\mathbb{W}_t$$

The Euler scheme for the transformed process gives

$$\Delta Y = Y_{i+1} - Y_i = \left(\frac{S(t_i, X_i)}{\sigma(t_i, X_i)} - \frac{1}{2}\sigma_x(t_i, X_i) \right) \Delta t + \sqrt{\Delta t} Z$$

Write the Taylor expansion for the inverse transformation $X = G(Y)$

$$G(Y + \Delta Y) = G(Y) + G'(Y)\Delta Y + \frac{1}{2} G''(Y)\Delta Y^2 + \mathcal{O}(\Delta Y^3)$$

where

$$G'(y) = \frac{1}{F'(G(y))} = \sigma(t, G(y))$$
$$G''(y) = G'(y)\sigma_x(t, G(y)) = \sigma(t, G(y))\sigma_x(t, G(y))$$

This gives

$$\begin{aligned} G(Y_i + \Delta Y) - G(Y_i) &= \text{(after some algebra)} \\ &= \left(S - \frac{1}{2}\sigma_x \sigma \right) \Delta + \sigma \sqrt{\Delta} Z + \frac{1}{2}\sigma\sigma_x \Delta Z^2 + \dots \\ &= \text{Milstein for the } X \text{ process} \end{aligned}$$

CIR process

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 \sqrt{X_t} d\mathbb{W}_t$$

Milstein scheme (after some re-ordering):

$$X_{i+1} = X_i + \left((\theta_1 - \theta_2 X_i) - \frac{1}{4}\theta_3^2\right)\Delta + \theta_3 \sqrt{X_i}\sqrt{\Delta} Z + \frac{1}{4}\theta_3^2 \Delta Z^2$$

Transform using $Y = F(X) = \sqrt{X}$. Itô formula gives

$$dY_t = \frac{1}{2Y_t}\left((\theta_1 - \theta_2 Y_t^2) - \frac{1}{4}\theta_3^2\right) dt + \frac{1}{2}\theta_3 d\mathbb{W}_t$$

for which the Euler scheme is

$$\Delta Y = \frac{1}{2Y_i}\left((\theta_1 - \theta_2 Y_i^2) - \frac{1}{4}\theta_3^2\right)\Delta + \frac{1}{2}\theta_3 \sqrt{\Delta} Z$$

CIR process

The inverse transformation is $x = G(y) = y^2$. Then

$$G(Y_i + \Delta Y) - G(Y_i) = (Y_i + \Delta Y)^2 - Y_i^2 = (\Delta Y)^2 + 2Y_i \Delta Y$$

Replace ΔY with the expression above, ignore high order terms, obtain

$$\begin{aligned} \Delta X &= G(Y_i + \Delta Y) - G(Y_i) \\ &= \left(\theta_1 - \theta_2 X_i - \frac{1}{4}\theta_3^2\right)\Delta + \theta_3 \sqrt{X_i}\sqrt{\Delta} Z + \frac{1}{4}\theta_3^2 \Delta Z^2 + \dots \end{aligned}$$

(Milstein scheme for the X process ...)

Predictor-corrector method

Tries to correct the fact that $S(t, X_t)$ and $\sigma(t, X_t)$ are not constant on the interval $[t_i, t_{i+1})$.
Step 1: predictor:

$$\tilde{X}_{i+1} = X_i + S(t_i, X_i)\Delta t + \sigma(t_i, X_i)\sqrt{\Delta t}Z;$$

Step 2: corrector:

$$\begin{aligned} X_{i+1} = X_i + &\Big(\alpha \tilde{S}(t_{i+1}, \tilde{X}_{i+1}) + (1-\alpha)\tilde{S}(t_i, X_i)\Big)\Delta t \\ &+ \Big(\eta\sigma(t_{i+1}, \tilde{X}_{i+1}) + (1-\eta)\sigma(t_i, X_i)\Big)\sqrt{\Delta t}Z \end{aligned}$$

where

$$\tilde{S}(t_i, X_i) = S(t_i, X_i) - \eta\sigma(t_i, X_i)\sigma_x(t_i, X_i)$$

This reduces to Euler method when $\alpha = \eta = 0$.

Second Milstein scheme

$$
\begin{aligned}
X_{i+1} = X_i &+ \left(S(t_i, X_i) - \frac{1}{2}\sigma(t_i, X_i)\sigma_x(t_i, X_i) \right) \Delta t + \sigma(t_i, X_i)\sqrt{\Delta t}\ Z \\
&+ \frac{1}{2}\sigma(t_i, X_i)\sigma_x(t_i, X_i)\Delta t\ Z^2 \\
&+ \Delta t^{3/2} \left(\frac{1}{2} S(t_i, X_i)\sigma_x(t_i, X_i) + \frac{1}{2} S_x(t_i, X_i)\sigma(t_i, X_i) + \frac{1}{4}\sigma(t_i, X_i)^2 \sigma_{xx}(t_i, X_i) \right)\ Z \\
&+ \Delta t^2 \left(\frac{1}{2} S(t_i, X_i) S_x(t_i, X_i) + \frac{1}{4} S_{xx}(t_i, X_i)\sigma(t_i, X_i)^2 \right)
\end{aligned}
$$

It has weak order of convergence $\beta = 2$.

Local linearization methods

Euler method assumes that the drift and diffusion coefficients and constant on small time intervals;
Assume that locally, the drift and diffusion coefficients are linear;
Ozaki method

$$dX_t = b(X_t)dt + \sigma d\mathbb{W}_t$$

Start with the corresponding deterministic system

$$\frac{dx_t}{dt} = b(x_t)$$

which admits the following numerical approximation

$$x_{t+\Delta t} = x_t + \frac{b(x_t)}{b_x(x_t)}\left(e^{b_x(x_t)\Delta t} - 1\right)$$

Assuming that $b(x) = K_t x$ on the interval $[t, t + \Delta t)$ gives

$$X_{t+\Delta t} = X_t e^{K_t \Delta t} + \sigma \int_t^{t+\Delta t} e^{K_t(t+\Delta t-u)} d\mathbb{W}_u$$

The constant K_t is determined from the assumption

$$E(X_{t+\Delta t} \mid X_t) = X_t e^{K_t \Delta t} = X_t + \frac{b(X_t)}{b_x(X_t)} \left(\exp\{b_x(X_t)\Delta t\} - 1\right)$$

(solve for K_t)

This gives

$$\mathcal{L}(X_{t+\Delta t} \mid X_t = x) = N(E_x, V_x)$$

where

$$E_x = x + \frac{b(x)}{b_x(x)} \left(\exp\{b_x(x)\Delta t\} - 1\right) \qquad V_x = \sigma^2 \frac{e^{2K_x \Delta t} - 1}{2K_x}$$

Shoji-Ozaki method

$$dX_t = b(t, X_t)dt + \sigma(X_t)d\mathbb{W}_t$$

Use the Lamperti transform to transform this SDE into

$$dX_t = b(t, X_t)dt + \sigma d\mathbb{W}_t$$

Use a "better" local approximation for $b(t, X_t)$ using first and second derivatives of $b(\cdot, \cdot)$. The corresponding discretization is given by

$$X_{t+\Delta t} = A(X_t)X_t + B(X_t)Z$$

where

$$A(X_s) = 1 + \frac{b(s, X_s)}{X_s L_s}(e^{L_s \Delta s} - 1) + \frac{M_s}{X_s L_s^2}(e^{L_s \Delta s} - 1 - L_s \Delta s)$$

$$B(X_s) = \sigma \sqrt{\frac{e^{2L_s \Delta s} - 1}{2L_s}}$$

Where

$$L_s = b_x(s, X_s) \quad M_s = \frac{1}{2} b_x x(s, X_s) + b_t(s, X_s);$$

Thus

$$\mathcal{L}(X_{t+\Delta t} \mid X_t = x) = N(A(x)x, B^2(x))$$

Asymptotics (LLN and CLT)

- In this section we present a few results about quantities like

$$\int_0^T h(X_t)dt \qquad \int_0^T h(X_t)d\mathbb{W}_t$$

  as $T \to \infty$.

- These are the continuous time versions of

$$\sum_{i=1}^{n} h(X_i)$$

- When appropriately normalized, we expect the "usual" limits, **if the process X^T is well-behaved**.
- Why do we care ? Many estimators (based on high-freq. data) are expressed using such quantities, thus, one can expect to establish asymptotic properties of these estimators.

Preliminaries

$$dX_t = S(X_t)dt + \sigma(X_t)d\mathbb{W}_t$$

Denote

$$\tau_a = \inf\{t \geq 0, X_t = a\} \qquad \tau_{ab} = \inf\{t \geq \tau_a : X_t = b\}$$

Definition The process X_t is called

- *recurrent* if $P(\tau_{ab} < \infty) = 1$;
- *positive recurrent* if $E(\tau_{ab}) < \infty$.
- *null recurrent* if $E(\tau_{ab}) = \infty$.

Proposition. The process X is recurrent if and only if

$$V(x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(u)}{\sigma(u)^2}\,du\right\}dy \to \pm\infty \tag{1}$$

as $x \to \pm\infty$. The recurrent process is positive if and only if

$$G = \int_{-\infty}^{\infty} \sigma(y)^{-2} \exp\left\{2\int_0^y \frac{S(u)}{\sigma(u)^2}\,du\right\}dy < \infty \tag{2}$$

The process is null recurrent if it is recurrent and

$$G = \infty$$

If $\sigma \equiv 1$ then (2) implies (1). In this case, the condition

$$\limsup_{|x|\to\infty} xS(x) < -1/2$$

is sufficient for (1) and (2).

Example

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 d\mathbb{W}_t \qquad \theta_2, \theta_3 > 0$$

Verify that

$$G = \int_{-\infty}^{\infty} \sigma(y)^{-2} \exp\left\{2\int_0^y \frac{S(u)}{\sigma(u)^2} du\right\} dy < \infty$$

It is also true that $V(x) \to \pm\infty$ as $x \to \pm\infty$. (exercise)

Ergodicity

The process X is ergodic if there exists an (invariant) distribution $F(\cdot)$ such that for any measurable $h(\cdot)$ such that $E(h(\xi)) < \infty$, we have the convergence

$$\frac{1}{T}\int_0^T h(X_t)dt \to E(h(\xi)) \quad \text{as} \quad T \to \infty \quad \text{a.s.}$$

Here we assume that $\xi \sim F(\cdot)$. From now on, we assume that

$(\mathcal{RP})$ $V(x) \to \pm\infty$ as $x \to \pm\infty$ and $G < \infty$.

Theorem. (Law of Large Numbers) Let the conditions $(\mathcal{RP})$ be fulfilled. Then, the process X is ergodic with invariant density given by

$$f(x) = \frac{1}{G\sigma(x)^2}\exp\left\{2\int_0^x \frac{S(y)}{\sigma(y)^2}dy\right\}$$

Example

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 d\mathbb{W}_t \quad \theta_2, \theta_3 > 0$$

The invariant density is

$$\begin{aligned} f(x) &\propto \exp\left\{ \frac{2}{\theta_3^2} \int_0^x (\theta_1 - \theta_2 y)\, dy \right\} \\ &= \exp\left\{ \frac{2}{\theta_3^2}(\theta_1 x - \theta_2 x^2/2) \right\} \\ &= \mathcal{N}\left(\text{mean } = \frac{\theta_1}{\theta_2} \ \text{ var } = \frac{\theta_3^2}{2\theta_2} \right) \end{aligned}$$

CLTs

Assume that $h \in \mathcal{M}_T^2$, that is $\int_0^T h(t, \omega)\ d\mathbb{W}_t$ is well defined.

Theorem. Say there exists a (non-random) function $\varphi(T)$ such that

$$\varphi(T)^2 \int_0^T h(t, \omega)^2\ dt \xrightarrow{\mathrm{P}} \rho^2 < \infty$$

Then,

$$\varphi(T) \int_0^T h(t, \omega)\ d\mathbb{W}_t \Rightarrow N(0, \rho^2)$$

(see Kutoyants, p.43)

CLT fo SDEs

$$dX_t = S(X_t)dt + \sigma(X_t)d\mathbb{W}_t$$

Assume that the $(\mathcal{R}\mathcal{P})$ conditions hold, i.e., (X_t) is positive recurrent and LLN holds

$$\frac{1}{T}\int_0^T g(X_t)^2 \; dt \xrightarrow{\mathrm{P}} \mathsf{E}(g(\xi)^2) \equiv \rho^2$$

It follows that

$$\frac{1}{\sqrt{T}}\int_0^T g(X_t) \; d\mathbb{W}_t \Rightarrow N(0, \rho^2)$$

More asymptotics

One can also formulate a CLT for the ordinary integral

$$\frac{1}{\sqrt{T}} \int_0^T h(X_t)dt$$

How ?

$$\frac{1}{\sqrt{T}} \int_0^T h(X_t)dt = \frac{H(X_T) - H_(X_0)}{\sqrt{T}} - \frac{1}{\sqrt{T}} \int_0^T H'(X_t)\sigma(X_t)d\mathbb{W}_t$$

where

$$H(x) = \int_0^x \frac{2}{\sigma(y)^2 f(y)} \int_{-\infty}^y h(v)f(v)dvdy$$

Statistical inference based on high frequency data

Assume that $X^T = \{X_t,\ t \in [0, T]\}$ is a diffusion process satisfying

$$dX_t = S(\theta, X_t)dt + \sigma(\theta, X_t)d\mathbb{W}_t \quad X(0) = X_0\ t \in [0, T]$$

Assume that we observe an **entire function** $\{X(t)\ t \in [0, T]\}$.

Goal: estimate θ.

Estimating the diffusion coefficient

Given $\{X(t)\ t \in [0, T]\}$, $\sigma(\theta, X_t)$ can be estimated with **high accuracy**, based on the quadratic variation. We have

$$\sum_{j=0}^{n-1} \Big(X(u_{j+1} - X(u_j)\Big)^2 \to \int_0^t \sigma(\theta, X_s)^2\, ds$$

where $0 = u_0 < u_1 < \cdots < u_n = t$ is a partition of $[0, t]$.

It follows that the RHS (the limit) can be “approximated” with any level of accuracy. Thus, the diffusion coefficient is determined.

Example: OU model

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 d\mathbb{W}_t \quad t \in [0, T]$$

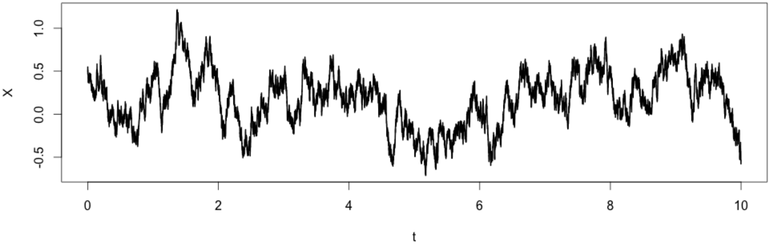
The quadratic variation of an observed path X^T is

$$\int_0^T \theta_3^2 ds = \theta_3^2 T \approx \sum_{i=1}^{N} (X(t_i) - X(t_{i-1}))^2$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a fine partition of $[0, T]$.
The RHS above can be evaluated with any level of precision, and thus one can estimate

$$\hat{\theta}_3^2 = \frac{1}{T} \sum_{i=1}^{N} (X(t_i) - X(t_{i-1}))^2$$

A short simulation



$\theta_1 = 1, \theta_2 = 2, \theta_3 = 1,\ T = 10$,
The estimate is $\hat{\theta}_3^2 = (1/T)\sum(X_{i+1} - X_i)^2$.

- $1,000$ discretization steps $\hat{\theta}_3 = 1.033$
- $10,000$ discretization steps $\hat{\theta}_3 = 1.006$

Estimating the drift coefficient

$$dX_t = S(\theta, X_t)dt + \sigma(X_t)d\mathbb{W}_t \quad X(0) = X_0 \ t \in [0, T]$$

Assume that we observe an **entire function** $X^T = \{X(t) \ t \in [0, T]\}$.
(note that no parameters appear in the diffusion coefficient)

We work under the assumption that X^T was generated by a **true** model, say

$$dX_t = S(5, X_t) + \sigma(X_t)d\mathbb{W}_t \quad X(0) = X_0 \ t \in [0, T] \quad ?$$

How do we estimate θ ?

Likelihood function

$$dX_t = S(\theta, X_t)dt + \sigma(X_t)d\mathbb{W}_t \quad X(0) = X_0 \ t \in [0, T] \qquad (3)$$

$$dX_t = S(\theta^*, X_t)dt + \sigma(X_t)d\mathbb{W}_t \quad X(0) = X_0 \ t \in [0, T] \qquad (4)$$

- The observed path $X^T = \{X_t, t \in [0, T]\}$ is an element of $\mathcal{C}([0, T])$ and is generated by model (4), $\theta^* =$truth.
- Model (3) induces a probability measure $\mathbb{Q}_\theta$ over $\mathcal{C}([0, T])$.
- Model (4) induces a probability measure $\mathbb{Q}_{\theta^*}$ over $\mathcal{C}([0, T])$.
- The likelihood function is

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{Q}_{\theta^*}}(X^T)$$

Recall the Girsanov formula

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{Q}_{\theta^*}}(X^T) = \exp\Big\{ \int_0^T \frac{S(\theta, X_t) - S(\theta^*, X_t)}{\sigma(X_t)^2}\ dX_t \\ - \frac{1}{2}\int_0^T \frac{S(\theta, X_t)^2 - S(\theta^*, X_t)^2}{\sigma(X_t)^2}\ dt \Big\}$$

The Maximum Likelihood Estimate (MLE) is then

$$\hat{\theta}_{MLE} = \mathrm{argmax}_\theta \frac{d\mathbb{Q}_\theta}{d\mathbb{Q}_{\theta^*}}(X^T)$$

Example: OU model

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 d\mathbb{W}_t$$
$$dX_t = (\theta_1^* - \theta_2^* X_t)dt + \theta_3 d\mathbb{W}_t$$

Data $X^T = \{X_t, t \in [0, T]\}$. Here $\theta = (\theta_1, \theta_2)$, $\theta^* = (\theta_1^*, \theta_2^*)$ and θ_3 is assumed known (from the quadratic variation). Maximize the likelihood function wrt θ. For this example, assume that θ_2 is also known, $\theta_2 = \theta_2^*$.

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{Q}_{\theta^*}}(X^T) = \exp\left\{ \int_0^T \frac{S(\theta, X_t) - S(\theta^*, X_t)}{\sigma(X_t)^2}\, dX_t - \frac{1}{2}\int_0^T \frac{S(\theta, X_t)^2 - S(\theta^*, X_t)^2}{\sigma(X_t)^2}\, dt \right\}$$

Note: $\sigma(X_t) = \theta_3$ and $S(\theta, X_t) = \theta_1 - \theta_2 X_t$.

Example

Ignore terms in θ^*, θ_3, maximize

$$\int_0^T S(\theta, X_t) dX_t - \frac{1}{2}\int_0^T S(\theta, X_t)^2 dt$$

Ignore terms in θ_2 (assumed known here), maximize

$$\int_0^T \theta_1 dX_t - \frac{1}{2}\left(t\theta_1^2 - 2\int_0^T \theta_1\theta_2 X_t dt\right)$$

Observe the quadratic, which is maximized at

$$\hat{\theta}_1^{MLE} = \frac{X_T - X_0 + \theta_2 \int_0^T X_t dt}{T}$$

Example: asymptotics

$$\begin{aligned}
\hat{\theta}_1^{MLE} &= \frac{X_T - X_0 + \theta_2 \int_0^T X_t dt}{T} \\
&= \frac{1}{T}\left(\int_0^T \theta_1^* - \theta_2^* X_t dt + \theta_3 \int_0^T d\mathbb{W}_t + \theta_2 \int_0^T X_t dt\right) \\
&= \theta_1^* + \frac{\theta_3}{T}\mathbb{W}_T
\end{aligned}$$

And thus

$$\sqrt{T}(\hat{\theta}_1^{MLE} - \theta_1^*) = \theta_3 \frac{\mathbb{W}_T}{\sqrt{T}} = N(0, \theta_3^2)$$

Exercises :

- assume that θ_1 is known and estimate θ_2
- assume that both (θ_1, θ_2) are unknown

Statistical inference based on discrete data

Consider a diffusion process

$$dX_t = S(\theta, X_t)dt + \sigma(\theta, X_t)d\mathbb{W}_t \quad X(0) = X_0 \ t \in [0, T]$$

Assume that we observe

$$\begin{aligned}\mathbf{X} &= (X(t_0), X(t_1), \ldots, X(t_n)) \\ &= (X_0, X_1, \ldots, X_n) \qquad \text{where } X_i = X(t_i)\end{aligned}$$

and $0 = t_0 < t_1 < \cdots < t_n = T$.

Goal: estimate θ.

Likelihood inference

Notation $[\,\cdot\,]$ – "distribution of ..." (typically referring to the pdf)

Inference on θ is based on the likelihood function

$$\begin{aligned} L_n(\theta) &= [\mathbf{X} \mid \theta] = [X_0, X_1, \ldots, X_n \mid \theta] \\ &= [X_0 \mid \theta] \prod_{i=1}^{n} [X_i \mid X_{i-1}, \theta] \qquad \text{(Markov process)} \end{aligned}$$

where $[X_i \mid X_{i-1}, \theta] =$ probability density of X_i given X_{i-1}, θ and t_{i-1}, t_i.
We use the general notation

$$p_\theta(x \mid \Delta, X_t = x_0) = \mathbb{P}_\theta(X_{t+\Delta} \in dx \mid X_t = x_0)$$

This quantity is called the **transition density**, and plays an extremely important role.

$$L_n(\theta) = p_\theta(X_0) \prod_{i=1}^{n} p_\theta(X_i \mid \Delta_i, X_{i-1})$$

Maximum likelihood estimation

- The MLE of θ is defined as

$$\hat{\theta}_{MLE} = argmax_{\theta} L_n(\theta) \quad \text{or} \quad \hat{\theta}_{MLE} = argmax_{\theta} \log(L_n(\theta))$$

- Exact likelihood inference can only be done in a handful of cases, where the transition density is known.

  (recall)

$$L_n(\theta) = p_{\theta}(X_0) \prod_{i=1}^{n} p_{\theta}(X_i \mid \Delta_i, X_{i-1})$$

OU model

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 d\mathbb{W}_t \qquad X(0) = X_0,\ \ t \in [0, T]\ \ \theta_2, \theta_3 > 0$$

The transition density is

$$p_\theta(x \,|\, X_t = x_0, \Delta) = \phi(x; m_\theta(x_0, \Delta), v_\theta(x_0, \Delta))$$

$\phi(x; \mu, \gamma^2)$ is the Gaussian pdf with mean μ, variace γ^2 evaluated at x.
Above,

$$m_\theta(x_0, \Delta) = \frac{\theta_1}{\theta_2} + \left(x_0 - \frac{\theta_1}{\theta_2}\right) e^{-\theta_2 \Delta}$$

$$v_\theta(x_0, \Delta) = \frac{\theta_3^2(1 - e^{-2\theta_2 \Delta})}{2\theta_2}$$

- Likelihood inference will work well, without much trouble
- In certain cases, once can even get closed form expressions for $\hat{\theta}_{MLE}$ ($\theta_1 = 0$)
- Asymptotic properties of $\hat{\theta}_{MLE}$ can also be obtained, typically one requires $n\Delta_n \to \infty$.

GBM model

$$dX_t = \theta_1 X_t dt + \theta_2 X_t d\mathbb{W}_t, \qquad X(0) = X_0, t \in [0, T] \;\; \theta_2 > 0$$

The transition distribution in this case is

$$p_\theta(x \mid X_t = x_0, \Delta) \sim \mathrm{log-Normal(mean = m_\theta(x_0, \Delta), var = v_\theta(x_0, \Delta))}$$

where

$$m_\theta(x_0, \Delta) = x_0 e^{\theta_1 \Delta}$$
$$v_\theta(x_0, \Delta) = x_0^2 e^{2\theta_1 \Delta} \left(e^{\theta_2^2 \Delta} - 1 \right)$$

CIR model

$$dX_t = (\theta_1 - \theta_2 X_t) dt + \theta_3 \sqrt{X_t} d\mathbb{W}_t \qquad X(0) = X_0, t \in [0, T] \; \theta_1, \theta_2, \theta_3 > 0$$

The transition density in this case is a non-central χ^2.

Pseudo-likelihood methods

- In most cases, the transition density is unknown and thus the likelihood function is intractable.
- In all these cases, one has to resort to a way to approximate the likelihood function.
- These pseudo-likelihood methods are of four types:
  - methods based on numerical discretizations of the SDE
  - methods based on a *simulated* likelihood
  - closed form approximations (not included in these notes)
  - methods based on exact sampling (EA) algorithms

Likelihood approximations based on the Euler scheme

$$dX_t = S(\theta, X_t)dt + \sigma(\theta, X_t)d\mathbb{W}_t \quad X(0) = X_0 \ t \in [0, T]$$

Consider the Euler approximation

$$X_{t+\Delta} - X_t = S(\theta, X_t)\Delta + \sigma(\theta, X_t)(\mathbb{W}_{t+\Delta} - \mathbb{W}_t)$$

Observing that $\mathbb{W}_{t+\Delta} - \mathbb{W}_t$ is independent of X_t, it follos that

$$X_{t+\Delta} \,|\, X_t, \theta \sim \mathcal{N}\Big(X_t + S(\theta, X_t), \quad \Delta\sigma(\theta, X_t)^2\Big)$$

The transition density is then approximated with

$$p_\theta(x \,|\, \Delta, X_0 = x_0) \approx \phi(x; m_\theta(x_0, \Delta), v_\theta(x_0, \Delta))$$

where

$$m_\theta(x_0, \Delta) = x_0 + S(\theta, x_0)\Delta \qquad v_\theta(x_0, \Delta) = \Delta\sigma(\theta, x_0)^2$$

$\phi(x; \mu, \gamma^2)$ is the Gaussian pdf with mean μ, variance γ^2 evaluated at x.

The effect of Δ

Consider the OU process

$$dX_t = (\theta_1 - \theta_2 X_t)dt + \theta_3 d\mathbb{W}_t \qquad X(0) = X_0,\ t \in [0, T]\ \ \theta_2, \theta_3 > 0$$

Recall that the **exact** transition density is Gaussian, with parameters

$$m_\theta(x_0, \Delta) = \frac{\theta_1}{\theta_2} + \left(x_0 - \frac{\theta_1}{\theta_2}\right) e^{-\theta_2 \Delta}$$
$$v_\theta(x_0, \Delta) = \frac{\theta_3^2(1 - e^{-2\theta_2\Delta})}{2\theta_2}$$

The **Euler** transition density is also Gaussian, with parameters

$$m_\theta(x_0, \Delta) = x_0(1 - \theta_2\Delta) + \theta_1\Delta$$
$$v_\theta(x_0, \Delta) = \theta_3^2\Delta$$

The two are "close", only as $\Delta \to 0$.

Simulated likelihood approximations

Focus again on the transition density and note that, informally

$$\begin{aligned}[X_\Delta \mid X_0] &= \int [X_\Delta, X_{\Delta-\delta} \mid X_0] \; dX_{\Delta-\delta} \\ &= \int [X_\Delta \mid X_{\Delta-\delta}][X_{\Delta-\delta} \mid X_0] dX_{\Delta-\delta} \\ &= \mathsf{E}\Big(p_\theta(X_\Delta \mid X_{\Delta-\delta}, \delta) \Big)\end{aligned}$$

where the expectation is taken wrt $X_{\Delta-\delta}$ (and X_Δ is held fixed).
Idea :

- If δ is v. small, then p_θ can be approximated with the Euler transition density
- The expectation above can be estimated using a Monte Carlo approach (sample many many $X_{\Delta-\delta}$ ...)

Importance sampling approach

Formally,

$$p_\theta(x \mid x_0, \Delta) =$$
$$= \int p_\theta(z_1, z_2, \dots, z_N \mid x_0, \delta) p_\theta(x \mid z_N, \delta) dz_1 dz_2 \dots dz_N$$
$$= \int \frac{p_\theta(z_1 \mid x_0, \delta) p_\theta(z_2 \mid z_1, \delta) \dots p(z_N \mid z_{N-1}, \delta) p_\theta(x \mid z_N, \delta)}{q(z_1, z_2, \dots, z_N)} q(z_1, z_2, \dots, z_N) dz_1 dz_2 \dots dz_N$$

where $q(\cdot)$ is some importance sampling density. The choice of $q(\cdot)$ is very important !

- The **main** idea is to make q depend on x and x_0 !!!!
- Elerian(2001) suggests a mutivariate Gaussian (or t) distribution
- Durham and Gallant (2002) propose Brownian Bridge samplers
- Stramer and Yan (2007) presents a good overview of this approach.

Approximations based on the exact sampling algorithms

EA (1,2 and 3) is a series of algorithms giving exact (no discretization error) draws from a large class of diffusion processes.
See Beskos and Roberts (2005), Beskos et al. (2006), Beskos et al. (2008)

EA algorithms are rejection sampling algorithms !

Recall the likelihood function

$$L_n(\theta) = p_\theta(X_0) \prod_{i=1}^{n} p_\theta(X_i \,|\, \Delta_i, X_{i-1})$$

Ideea: given $X_{i-1}, X_i, \Delta, \theta$, one can simulate a random variable Ψ such that

$$\mathrm{E}(\Psi) = p_\theta(X_i \,|\, X_{i-1}, \Delta)$$

Approximations based on the exact sampling algorithms

That is, each contribution to the likelihood function can be estimated unbiasedly: simulate iid $\Psi_1, \ldots, \Psi_B$ and estimate

$$\hat{p}_\theta(X_i \,|\, X_{i-1}, \Delta) = \frac{1}{B} \sum_{j=1}^{B} \Psi_j$$

This leads to an unbiased estimate of the likelihood function, which can be consequently optimized over θ.

Why EA ? The idea is that

$$p_\theta(X_i \,|\, X_{i-1}, \Delta) = \mathrm{E}(\text{"acceptance probability from EA algorithm"})$$